Substitution Rule

If $u = g(x)$ is differentiable function whose range is an Interval I + f = continuous on I, then
$\int f(g(x))g'(x)\,dx = \int f(u)\,du$

$\int 3x^2 e^{x^3} dx$ $\quad u = x^3 \quad du = 3x^2 dx$

$\int e^u du = e^u + c = e^{x^3} + c$

$u = e^{x^3} \quad du = e^{x^3} 3x^2 dx$

$\int du = u + c = e^{x^3} + c$

$\int \frac{\cos(\theta)}{\sin^2(\theta)} d\theta = \int \frac{du}{u^2}$ $\quad \cancel{u = \cos(\theta)} \quad du = -\sin(\theta)d\theta$

$= \int u^{-2} du = \frac{u^{-1}}{-1} + c$ $\quad u = \cancel{\sin^2}(\theta) \quad du = 2\sin(\theta)\cos(\theta)d\theta$

$= -\frac{1}{\sin(\theta)} + c$ $\quad u = \sin(\theta) \quad du = \cos(\theta)d\theta$

$\int 5x\sqrt{3x^2 - 2}\,dx$ $\quad u = 3x^2 - 2 \quad \frac{du}{6} = \frac{\cancel{6}x\,dx}{\cancel{6}}$

$5\int \sqrt{u}\,\frac{du}{6} = \frac{5}{6}\int u^{1/2} du = \frac{5}{6}\frac{u^{3/2}}{3/2} + c = \frac{5}{6}\frac{2}{3}u^{3/2} + c$

$= \frac{5}{9}u^{3/2} + c = \frac{5}{4}(3x^2 - 2)^{3/2} + c$

If g' is continuous on $[a,b]$ + f = continuous on range of $u = g(x)$, then $\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$

$\int_0^{\frac{\pi}{4}} \tan(\theta)d\theta = \int_0^{\frac{\pi}{4}} \frac{\sin(\theta)}{\cos(\theta)} d\theta$ $\quad u = \cos(\theta) \quad du = -\sin(\theta)d\theta$

$\theta = 0,\ u = 1 \quad \theta = \frac{\pi}{4},\ u = \frac{\sqrt{2}}{2}$

$= \int_1^{\frac{\sqrt{2}}{2}} -\frac{1}{u} du = -\ln|u| \Big|_1^{\sqrt{2}/2}$

$= -\ln\left|\frac{\sqrt{2}}{2}\right| - (-\ln|1|) = -\ln\left(\frac{\sqrt{2}}{2}\right) = \ln(\sqrt{2})$

Substitution Rule

If $u = g(x)$ is differentiable function whose range is an Interval I + f = continuous on I, then

$\int f(g(x))g'(x)\,dx = \int f(u)\,du$

$\int 3x^2 e^{x^3} dx$ $\qquad u = x^3 \qquad du = 3x^2 dx$

$\int e^u du = e^u + c = e^{x^3} + c$

$u = e^{x^3} \qquad du = e^{x^3} 3x^2 dx$

$\int du = u + c = e^{x^3} + c$

$\int \frac{\cos(\theta)}{\sin^2(\theta)} d\theta = \int \frac{du}{u^2}$ $\qquad u = \cancel{\cos(\theta)} \qquad du = -\sin(\theta)d\theta$

$= \int u^{-2} du = \frac{u^{-1}}{-1} + c$ $\qquad u = \cancel{\sin}^2(\theta) \qquad du = 2\sin(\theta)\cos(\theta)d\theta$

$= -\frac{1}{\sin(\theta)} + c$ $\qquad u = \sin(\theta) \qquad du = \cos(\theta)d\theta$

$\int 5x\sqrt{3x^2 - 2}\, dx$ $\qquad u = 3x^2 - 2 \qquad \frac{du}{6} = \frac{\cancel{6}x\,dx}{\cancel{6}}$

$5\int \sqrt{u}\, \frac{du}{6} = \frac{5}{6}\int u^{1/2} du = \frac{5}{6} \frac{u^{3/2}}{3/2} + c = \frac{5}{6}\frac{2}{3} u^{3/2} + c$

$= \frac{5}{9} u^{3/2} + c = \frac{5}{9}(3x^2 - 2)^{3/2} + c$

If g' is continuous on $[a,b]$ + f = continuous on range of $u = g(x)$, then $\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$

$\int_0^{\frac{\pi}{4}} \tan(\theta)d\theta = \int_0^{\frac{\pi}{4}} \frac{\sin(\theta)}{\cos(\theta)} d\theta$ $\qquad u = \cos(\theta) \qquad du = -\sin(\theta)d\theta$

$\theta = 0,\ u = 1 \qquad \theta = \frac{\pi}{4},\ u = \frac{\sqrt{2}}{2}$

$= \int_1^{\frac{\sqrt{2}}{2}} -\frac{1}{u} du = -\ln|u| \Big|_1^{\sqrt{2}/2}$

$= -\ln\left|\frac{\sqrt{2}}{2}\right| - \left(-\ln|1|\right) = -\ln\left(\frac{\sqrt{2}}{2}\right) = \ln(\sqrt{2})$

$$\int_0^1 x^3\sqrt{1+x^2}\,dx \qquad u=1+x^2 \qquad du=2x\,dx$$

$$\int_0^1 x^2\sqrt{1+x^2}\,x\,dx \qquad x^2=u-1 \qquad \frac{du}{2}=x\,dx$$

$$x=0,\ u=1 \qquad x=1,\ u=2$$

$$\int_1^2 (u-1)\sqrt{u}\,\frac{du}{2} = \frac{1}{2}\int_1^2 \left(u^{3/2}-u^{1/2}\right)du = \frac{1}{2}\left(\frac{u^{5/2}}{5/2}-\frac{u^{3/2}}{3/2}\right)\Big|_1^2$$

$$= \left(\frac{1}{5}u^{5/2}-\frac{1}{3}u^{3/2}\right)\Big|_1^2$$

$$= \left(\frac{1}{5}(2)^{5/2}-\frac{1}{3}(2)^{3/2}\right)-\left(\frac{1}{5}(1)^{5/2}-\frac{1}{3}(1)^{3/2}\right)$$

$$= \frac{1}{5}4\sqrt{2}-\frac{1}{3}2\sqrt{2}-\frac{1}{5}+\frac{1}{3} = \frac{4\sqrt{2}}{5}-\frac{2\sqrt{2}}{3}-\frac{1}{5}+\frac{1}{3} = \frac{4\sqrt{2}}{5}-\frac{2\sqrt{2}}{3}+\frac{2}{15}$$

$$\int \frac{1+x}{1+x^2}\,dx \qquad u=1+x^2 \qquad du=2x\,dx \rightarrow \frac{du}{2}=x\,dx$$

$$\int\left(\frac{1}{1+x^2}+\frac{x}{1+x^2}\right)dx = \int\frac{1}{1+x^2}dx+\int\frac{x}{1+x^2}dx$$

$$= \tan^{-1}(x)+c_1+\frac{1}{2}\int\frac{du}{u} = \tan^{-1}(x)+c_1+\frac{1}{2}\ln|u|+c_2$$

$$= \tan^{-1}(x)+c_1+\frac{1}{2}\ln|1+x^2|+c_2 = \boxed{\tan^{-1}(x)+\frac{1}{2}\ln(1+x^2)+C}$$

Always >0, so do not need abs val

$$\int x^3\sec^2(x^4)\,dx \qquad u=x^4 \qquad du=4x^3\,dx$$

$$\frac{du}{4}=x^3\,dx$$

$$\frac{1}{4}\int\sec^2(u)\,du$$

$$\frac{1}{4}\tan(u)+c = \frac{1}{4}\tan(x^4)+c$$

$$\int\frac{1}{x\sqrt{\ln(x)}}\,dx \qquad u=\ln(x) \qquad du=\frac{1}{x}dx$$

$$\int\frac{1}{\sqrt{u}}\,du = \int u^{-1/2}\,du = \frac{u^{1/2}}{1/2}+c = 2\sqrt{\ln(x)}+c$$